The spin-statistics connection in classical field theory

2006 J. Phys. A: Math. Gen. 39 13337

(http://iopscience.iop.org/0305-4470/39/42/009)

J. Phys. A: Math. Gen. **39** (2006) 13337–13353 doi:10.1088/0305-4470/39/42/009

The spin-statistics connection in classical field theory

J A Morgan

The Aerospace Corporation, PO Box 92957, Los Angeles, CA 90009, USA

Received 12 May 2006, in final form 1 September 2006
Published 4 October 2006
Online at stacks.iop.org/JPhysA/39/13337

Abstract
The spin-statistics connection is obtained for a simple formulation of a classical field theory containing even and odd Grassmann variables. To that end, the construction of irreducible canonical realizations of the rotation group corresponding to general causal fields is reviewed. The connection is obtained by imposing local commutativity on the fields and exploiting the parity operation to exchange spatial coordinates in the scalar product of classical field evaluated at one spatial location with the same field evaluated at a distinct location. The spin-statistics connection for irreducible canonical realizations of the Poincaré group of spin j is obtained in the form: classical fields and their conjugate momenta satisfy fundamental field-theoretic Poisson bracket relations for $2j$ even and fundamental Poisson antibracket relations for $2j$ odd.

PACS numbers: 03.30+p, 03.50.−z

1. Introduction

Few would dispute that the field concept has been one of the happiest conceptual innovations in the history of physics. From its nineteenth-century origins in the work of Faraday and Maxwell to modern theories of fluid physics, the classical theory of elastic solids, relativistic gravitation, or of quantum field theory, the notion of a physical quantity which takes on values throughout a plenum has proven to be protean and endlessly fruitful. Consider the success of the very prototype of a field theory, classical electrodynamics, in describing the propagation of electromagnetic radiation, in identifying that radiation with light, and in accounting for the transport of electromagnetic energy by light with the Poynting vector. Imagine the difficulty of describing any of these phenomena without the aid of electromagnetic fields. Nor is the future development of quantum-mechanical descriptions of nature likely to abandon the field concept: whatever form a final theory of physics takes, whether cast in terms strings or branes, or other entities, 'effective' quantum field theories valid at low energies—the definition of 'low' varying, as occasion demands—will remain indispensable aids to practical calculation.

This paper presents a proof of the connection between spin and statistics for classical Grassmann fields. An earlier paper [1] presented a classical analogue of the spin-statistics connection for pseudomechanics, a version of analytical dynamics containing even and odd

Grassmann variables [2–7]. The method of proof relies on the canonical formalism for fields, suitably extended to include odd Grassmann variables. Classical field theory is readily cast in terms of the canonical formalism [8–10], most familiarly as part of many an introductory account of quantum field theory [11, 12]. (But see [13, 14] for a different account of the foundations of quantum field theory.) The literature on canonical formulations of classical field theory, studied in their own right, is too large and various for a capsule summary. A sampler of this body of research may be gleaned from reviews in [15–18]. On the other hand, apart from investigations inspired by supersymmetry [19, 20], classical treatments of odd Grassmann fields do not appear to be common. Examples may be found in papers by Gozzi *et al* [21] and Floreanini and Jackiw [22].

We begin by extending the canonical formalism for fields to classical Grassmann variables. The classical equivalent of fermionic exchange symmetry appears in the properties of anticommuting Grassmann variables. The construction of irreducible canonical realizations for massive fields possessing definite intrinsic spin, starting from the Lie algebra of the Poincaré group, is outlined next. Irreducible canonical realizations of the Poincaré group are classified in the same manner as irreducible unitary representations in quantum field theory. Canonical equivalents exist for the elements of the theory of unitary representations in a Hilbert space, including ladder and Casimir operators. Of particular importance for the present problem, the canonical space inversion (or parity) operation $\hat{P}$ and its action are defined. Finally, the spin-statistics connection is obtained by using the parity operation to exchange spatial coordinates in the scalar product of a field evaluated at one spacetime location with the same field, evaluated at a distinct spacetime location lying at spacelike interval from the first.

Local Poincaré symmetry contributes three elements to the proof: (1) local commutativity, (2) the properties of the rotational subgroup, specifically the properties of irreducible canonical realizations of spin degrees of freedom and (3) the action of the discrete symmetry of parity transformation $\hat{P}$. As in other proofs of the spin-statistics connection, invocation of Poincaré symmetry furnishes a sufficient condition for the connection. Its necessity is addressed in section 6.

Lower case Greek letters denote either even or odd Grassmann fields, unless otherwise indicated in the text. When it is desirable to distinguish even fields, they will be labelled by Latin letters of either case. Lower case Latin letters are also used for coordinates of spacetime locations; $x \equiv (\mathbf{x}, t)$ distinguishes space and time coordinates once a spacelike foliation has been established. Except when serving for spin degrees of freedom, lower case Greek indices run from 0 to 3, while lower case Latin indices run from 1 to 3. The summation convention applies to repeated indices, unless the summation sign is explicitly shown for emphasis. When required for notational compactness, the partial derivative of ψ_α with respect to x^μ is written as

$$\psi_{\alpha,\mu} \equiv \frac{\partial \psi_\alpha}{\partial x^\mu}. \tag{1}$$

2. Classical Grassmann fields

The commutation properties of Grassmann variables permit the realization of fermionic and bosonic exchange symmetry in a classical setting [1, 23, 24]. The familiar c-number bosonic variables of traditional classical physics are *even* Grassmann variables. *Odd* classical Grassmann variables anticommute in a form of the exclusion principle. A set of n odd real Grassmann fields obeys the anticommutation relations

$$\xi_\mu(x)\xi_\nu(y) + \xi_\nu(y)\xi_\mu(x) = 0. \qquad (\mu, \nu \leqslant n). \tag{2}$$

It follows that

$$\xi_\mu^2(x) = 0. \tag{3}$$

One even and one odd variable, in either order, commute. Differentiation on Grassmann variables can act from the right or the left. The sign of the derivative of a product, for example, can depend on which derivative is taken. Left differentiation, in accord with the convention in [23], is used exclusively in the following.

The formalism for analytical dynamics of Grassmann variables, developed by Casalbuoni, Pauri, Prosperi and Loinger, that found use in [1] can largely be translated directly into field-theoretic language [9, 10, 17, 25]. Canonical transformations, for example, are defined in a manner that corresponds closely to analytical dynamics, as automorphisms of the fields that preserve Poisson brackets. The necessary alterations to the definition of Poisson brackets for fields are given below. The discussion will be limited to unconstrained systems for simplicity.

Fields are taken to be (Poincaré-symmetric) elements of the Hilbert space of complex square-integrable functions. The canonical realizations of the Poincaré group to be described below form invariant subspaces of the Hilbert space. Fields are assumed to be 'massive,' so that it makes sense to speak of a rest frame for them, i.e., a frame in which the spatial components of the four-momentum density vanish[1]. The support of the fields is taken to be a large region of Minkowski spacetime R with boundary ∂R on which normal gradients of the fields vanish. Let $q_i(x^\mu)$, $i = 1, \ldots, m$, be even field variables, and $\xi_\alpha(x^\mu)$, $\alpha = 1, \ldots, n$, be odd field variables. Given a Lagrangian

$$L = L\left(q_i, \xi_\alpha, \frac{\partial q_i}{\partial x^\mu}, \frac{\partial \xi_\alpha}{\partial x^\mu}\right), \tag{4}$$

define the Lagrangian density by

$$L = \int \mathrm{d}^3x\, \mathcal{L}\left(q_i, \xi_\alpha, \frac{\partial q_i}{\partial x^\mu}, \frac{\partial \xi_\alpha}{\partial x^\mu}\right). \tag{5}$$

The Euler–Lagrange equation for a field ψ is

$$\frac{\delta L}{\delta \psi} = 0 \tag{6}$$

with

$$\frac{\delta L}{\delta \psi} \equiv \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} \tag{7}$$

for ψ even or odd. Generalized momenta are defined by

$$p^i = \frac{\delta L}{\delta q_{i,0}} \qquad \pi^\alpha = \frac{\delta L}{\delta \xi_{\alpha,0}}. \tag{8}$$

The Hamiltonian is given by

$$H = \int \mathrm{d}^3x \mathcal{H} \tag{9}$$

with

$$\mathcal{H} \equiv q_{i,0}p^i + \xi_{\alpha,0}\pi^\alpha - \mathcal{L}, \tag{10}$$

and Hamilton's equations are

$$q_{i,0} = \frac{\delta H}{\delta p^i} \qquad p^i_{,0} = -\frac{\delta H}{\delta q_i} \tag{11}$$

[1] This condition is presumably not necessary for the existence of a spin-statistics connection for a classical field, but it greatly simplifies any discussion of the effects of space inversion.

$$\xi_{\alpha,0} = -\frac{\delta H}{\delta \pi^{\alpha}} \qquad \pi^{\alpha}_{,0} = -\frac{\delta H}{\delta \xi_{\alpha}}. \tag{12}$$

By the way of introducing the Poisson bracket for Grassmann fields, let

$$F = \int \mathrm{d}^3x\, \mathcal{F}(q_i, q_{i,k}, \xi_\alpha, \xi_{\alpha,k}, p^i, \pi^\alpha) \tag{13}$$

be an even functional and consider its total rate of change

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}t} &= \int \mathrm{d}^3x \frac{\partial \mathcal{F}}{\partial t} + \int \mathrm{d}^3x \left\{ \frac{\partial \mathcal{F}}{\partial q_i} q_{i,0} + \frac{\partial \mathcal{F}}{\partial q_{i,k}} q_{i,k0} \right. \\ &\quad \left. + \frac{\partial \mathcal{F}}{\partial p^i} p^i_{,0} + \xi_{\sigma,0} \frac{\partial \mathcal{F}}{\partial \xi_\sigma} + \xi_{\sigma,k0} \frac{\partial \mathcal{F}}{\partial \xi_{\sigma,k}} + \pi^\sigma_{,0} \frac{\partial \mathcal{F}}{\partial \pi^\sigma} \right\}.\end{aligned} \tag{14}$$

A standard calculation, [26] modified as needed for the odd variables, gives

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \int \mathrm{d}^3x \frac{\partial \mathcal{F}}{\partial t} + \int \mathrm{d}^3x \left\{ \frac{\delta F}{\delta q_i}\frac{\delta H}{\delta p^i} - \frac{\delta F}{\delta p^i}\frac{\delta H}{\delta q_i} - \frac{\delta H}{\delta \pi^\alpha}\frac{\delta F}{\delta \xi_\alpha} - \frac{\delta H}{\delta \xi_\alpha}\frac{\delta F}{\delta \pi^\alpha} \right\}. \tag{15}$$

Rearranging (15) gives the template for the Poisson bracket for two even functionals of the field variables [2, 3]:

$$\begin{aligned}[F, G] &= \int \mathrm{d}^3x \left\{ \frac{\delta F}{\delta q_i}\frac{\delta G}{\delta p^i} - \frac{\delta G}{\delta q_i}\frac{\delta F}{\delta p^i} \right\} + \int \mathrm{d}^3x \left\{ \frac{\delta F}{\delta \xi_\alpha}\frac{\delta G}{\delta \pi^\alpha} - \frac{\delta G}{\delta \xi_\alpha}\frac{\delta F}{\delta \pi^\alpha} \right\} \\ &= -[G, F].\end{aligned} \tag{16}$$

As in [2], the definition of the remaining brackets is fixed by requiring that multiplication of fields by an odd Grassmann constant gives an algebra over the ring of Grassmann fields. The bracket of two odd functionals θ and ψ is given by

$$\begin{aligned}[\theta, \psi] &= \int \mathrm{d}^3x \left\{ \frac{\delta\theta}{\delta q_i}\frac{\delta\psi}{\delta p^i} + \frac{\delta\psi}{\delta q_i}\frac{\delta\theta}{\delta p^i} \right\} - \int \mathrm{d}^3x \left\{ \frac{\delta\theta}{\delta \xi_\alpha}\frac{\delta\psi}{\delta \pi^\alpha} + \frac{\delta\psi}{\delta \xi_\alpha}\frac{\delta\theta}{\delta \pi^\alpha} \right\} \\ &= [\psi, \theta]\end{aligned} \tag{17}$$

and is called an antibracket. When it is desired to emphasize the difference between brackets of two even variables and antibrackets, they will be written as $[F, G]^-$ and $[\theta, \psi]^+$, respectively. For an odd and an even functional

$$[\theta, F] = \int \mathrm{d}^3x \left\{ \frac{\delta\theta}{\delta q_i}\frac{\delta F}{\delta p^i} - \frac{\delta F}{\delta q_i}\frac{\delta\theta}{\delta p^i} \right\} - \int \mathrm{d}^3x \left\{ \frac{\delta\theta}{\delta \xi_\alpha}\frac{\delta F}{\delta \pi^\alpha} + \frac{\delta F}{\delta \xi_\alpha}\frac{\delta\theta}{\delta \pi^\alpha} \right\}. \tag{18}$$

The bracket between an even and an odd functional is defined so that

$$[F, \theta] = -[\theta, F]. \tag{19}$$

Casalbuoni [2] has shown that the set of Poisson brackets and antibrackets in pseudomechanics comprises a graded Lie algebra [27, 28]. The development just given recapitulates that construction in terms of field-theoretic brackets. Let $\varsigma_f = 0$ for an even field and $\varsigma_\phi = 1$ for odd. The corresponding generalization of the Jacobi identity, [2, 3, 27–29]

$$(-1)^{\varsigma_\pi\varsigma_\gamma}[\gamma, [\rho, \pi]] + (-1)^{\varsigma_\gamma\varsigma_\rho}[\rho, [\pi, \gamma]] + (-1)^{\varsigma_\rho\varsigma_\pi}[\pi, [\gamma, \rho]] = 0, \tag{20}$$

required for the construction of canonical angular momentum ladder operators in section 3.2 is cumbersome to prove using elementary methods. Given the validity of (20) in psuedomechanics, its validity for classical fields follows from results obtained by Kupershmidt in a treatment of classical fields possessing Hamiltonian structure, using a geometric formulation of the calculus of variations [19, 20, 25]. Jacobi identities for classical fields are not guaranteed to vanish under all circumstances. However, the identities do vanish modulo a trivial divergence, whose contribution in the present discussion vanishes upon integration over R, by virtue of the assumption that all gradients vanish on ∂R.

Fundamental Poisson brackets may be computed by making the substitution [30]

$$\xi(\mathbf{x},t) = \int \mathrm{d}^3z\, \xi(\mathbf{z},t)\delta(\mathbf{z}-\mathbf{x}), \tag{21}$$

allowing one to regard ξ formally as a functional for the purpose of permitting differentiation under the integral sign. For two even fields we find

$$[q_\mu(t,\mathbf{x}), q_\nu(t,\mathbf{y})]^- = [p^\mu(t,\mathbf{x}), p^\nu(t,\mathbf{y})]^- = 0. \tag{22}$$

The bracket between a field and its corresponding canonical momentum is computed with the aid of the identity [31]

$$\delta(\mathbf{x}-\mathbf{y}) = \int \mathrm{d}^3z\, \delta(\mathbf{x}-\mathbf{z})\delta(\mathbf{z}-\mathbf{y}), \tag{23}$$

leading to

$$[q_\mu(t,\mathbf{x}), p^\nu(t,\mathbf{y})]^- = \delta(\mathbf{x}-\mathbf{y})\delta_\mu^\nu. \tag{24}$$

For odd fields and momenta

$$[\xi_\mu(t,\mathbf{x}), \xi_\nu(t,\mathbf{y})]^+ = [\pi^\mu(t,\mathbf{x}), \pi^\nu(t,\mathbf{y})]^+ = 0 \tag{25}$$

and

$$[\xi_\mu(t,\mathbf{x}), \pi^\nu(t,\mathbf{y})]^+ = -\delta(\mathbf{x}-\mathbf{y})\delta_\mu^\nu. \tag{26}$$

3. Irreducible canonical realizations of the Poincaré group

The classical model of a field used in this paper is the canonical field-theoretic counterpart of an irreducible representation of the Poincaré group used to describe a particle in quantum field theory [32, 33]. Instead of commutation relations amongst matrix generators of infinitesimal Lorentz transformations and rotations, one manipulates Poisson brackets relating infinitesimal generators of canonical transformations. Likewise, a functional on phase space which depends solely upon the generators of the Lie algebra and which is an invariant in all realizations of the Lie group is called a Casimir invariant, or simply a Casimir. Casimirs serve as the canonical equivalents of quantum-mechanical Casimir operators.

Pauri and Prosperi [34] developed the theory of canonical realizations of Lie groups and later presented the canonical realization of the Poincaré group in detail [35] (*vide* also [36]). Morgan [1] gives the extension of that analysis to canonical realizations in pseudomechanics. Classical fields of the simple kind considered in this paper satisfy Poisson bracket relations identical to those for generalized coordinates and momenta in pseudomechanics. Insofar as the development in [35] depends only on algebraic relations of generators of canonical transformations and Poisson brackets, the results of that analysis can be translated into the corresponding field-theoretic results.

3.1. The Poincaré group

The realization of the Poincaré group in particle mechanics as a set of canonical transformations is presented in [1]. The corresponding development for fields is broadly analogous. The effect of a general inhomogeneous Lorentz transformation (Λ, a) on a 4-vector x_μ is

$$x'^\mu = \Lambda^\mu_\alpha x^\alpha + a^\mu. \tag{27}$$

If such a transformation is given a unitary representation (of any tensorial rank), the matrices D of the representation satisfy

$$D(\Lambda_2, a_2)D(\Lambda_1, a_1) = D(\Lambda_2\Lambda_1, a_2 + \Lambda_2 a_1). \tag{28}$$

Sufficiently near to the origin,

$$\Lambda^{\mu}_{\alpha} = \delta^{\mu}_{\alpha} + \omega^{\mu}_{\alpha} + O(\omega^2) \tag{29}$$

$$a^{\mu} = \epsilon^{\mu}, \tag{30}$$

with $|\omega|$ and $|\epsilon| \ll 1$. The quantity ω is antisymmetric in its indices. To first order, a representation of (29) and (30) is

$$D(1+\omega, \epsilon) = 1 + \tfrac{1}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu} - \epsilon_{\rho}\mathcal{P}^{\rho}, \tag{31}$$

where for each pair (μ, ν) $\mathcal{M}^{\mu\nu} = -\mathcal{M}^{\nu\mu}$ is a matrix generator of generalized rotations and $\mathcal{P}^{\mu}$ is a matrix generator of translations.

The commutation relations relating $\mathcal{M}$ and $\mathcal{P}$ comprise the Lie algebra of the Poincaré group [33, 36–38]. In classical field theory, one may exploit the matrix representation directly as in quantum field theory, or one may regard the commutation relations of the generators as determining Lie bracket relations for an abstract representation of the Poincaré group. We follow the latter path here and obtain a canonical realization by replacing Lie brackets with Poisson brackets for the generators M and P of infinitesimal canonical transformations corresponding to $\mathcal{M}$ and $\mathcal{P}$, as described in [1].

The ten generators of infinitesimal transformations for the Poincaré group can be sorted into one temporal translation, three spatial translations, three boosts and three rotations [33, 35, 36, 38]. Of these, only the generators of rotations and boosts,

$$\mathcal{J}_i = \epsilon_{ijk}\mathcal{M}^{jk} \qquad \mathcal{K}_i = \mathcal{M}^{i0} \qquad (i, j, k = 1\text{–}3), \tag{32}$$

with commutators

$$\mathcal{J}_i\mathcal{J}_j - \mathcal{J}_j\mathcal{J}_i = \epsilon_{ijk}\mathcal{J}_k \tag{33}$$

$$\mathcal{K}_i\mathcal{K}_j - \mathcal{K}_j\mathcal{K}_i = -\epsilon_{ijk}\mathcal{J}_k \tag{34}$$

$$\mathcal{J}_i\mathcal{K}_j - \mathcal{K}_j\mathcal{J}_i = \epsilon_{ijk}\mathcal{K}_k \tag{35}$$

play a role in the subsequent development.

A field-theoretical realization of the Poincaré algebra can be constructed in more than one way[2]; for illustrative purposes consider generators of canonical transformations derived from irreducible representations of the Poincaré group that are bilinear in the fields and canonical momenta [39, 40]. Given a matrix generator $\mathcal{G}_{\mu}(\mathbf{x}, \boldsymbol{\nabla}_x)$ of infinitesimal transformations for an irreducible representation of the Poincaré group, one may obtain a canonical generator G_{μ} in the form

$$G_{\mu} = \int \mathrm{d}^3x\,(p^i\mathcal{G}_{\mu}q_i - \pi^{\alpha}\mathcal{G}_{\mu}\xi_{\alpha}). \tag{36}$$

As in pseudomechanics, the canonical realization of the Poincaré group possesses no nontrivial neutral elements. The Poisson bracket relations of the canonical realization are thus identical to those of the Lie algebra. The canonical generators obtained by inserting (35) into (36)

[2] Finite-dimensional irreducible realizations of the Poincaré group can possess field components which do not possess conjugate momenta and which are not independent of other components, *vide* [10], pp 495–6. Since these additional components do not satisfy fundamental Poisson brackets, they are of no concern in the present connection.

satisfy the relations

$$[J_i, J_j]^- = \epsilon_{ijk} J_k \tag{37}$$

$$[K_i, K_j]^- = -\epsilon_{ijk} J_k \tag{38}$$

$$[J_i, K_j]^- = \epsilon_{ijk} K_k. \tag{39}$$

The realization of the Poincaré group so obtained consists of real generators.

The effect of an infinitesimal canonical transformation induced by a Lorentz transformation is given by

$$\delta\xi_\mu = \delta\alpha_\rho [G_\rho, \xi_\mu] \tag{40}$$

$$\delta\pi^\mu = \delta\alpha_\rho [G_\rho, \pi^\mu], \tag{41}$$

where G_ρ is one of the Lorentz generators and the constants $\delta\alpha_\rho$ are the Lorentz transformation parameters. The effect of a finite transformation on a field is given formally by exponentiating the action of (40) [41]:

$$\xi'_\mu(x'_\nu) = \exp(\alpha_\rho [G_\rho, \ldots])\xi_\mu(x_\lambda). \tag{42}$$

In the case of an irreducible realization, the result may also be expressed as

$$\xi'_\mu(x'_\nu) = D^\rho_\mu(\Lambda)\xi_\rho\big(\Lambda^\lambda_\nu x_\lambda + a_\nu\big), \tag{43}$$

where the matrix $D^\rho_\mu(\Lambda)$ belongs to the corresponding irreducible representation of the Lorentz group [42].

3.2. Angular momentum and irreducible canonical realizations

The behaviour under rotations of a field that transforms as an irreducible canonical realization with a definite angular momentum closely resembles that of a corresponding quantum-mechanical system. Canonical angular momentum variables satisfy bracket relations that form a subalgebra of the Poincaré algebra decoupled from the boost degrees of freedom. As a result, the spin degrees of freedom of a massive field in its rest frame are treated in the same way as in nonrelativistic quantum mechanics.

Suppose that a field is an irreducible canonical realization possessing definite angular momentum. Fix the direction of the z-axis along its spatial part and write ξ for the magnitude. The change in the value of ξ induced by an infinitesimal rotation about z is

$$\xi(x) \Rightarrow \xi(x) + \delta\phi[\xi, J_z]. \tag{44}$$

If ξ has rotational symmetry about the axis defining ϕ, the effect of this transformation must be equivalent to multiplication by a phase

$$\xi \Rightarrow \xi + \mathrm{i}m\delta\phi\xi, \tag{45}$$

or

$$[J_z, \xi] = -\mathrm{i}m\xi. \tag{46}$$

One may regard this relation as a kind of eigenvector condition and label ξ by its eigenvalue m as ξ_m [43]. The angular momentum ladder operators

$$J_\pm = J_x \pm \mathrm{i}J_y, \tag{47}$$

satisfy

$$[J_+, J_-]^- = -2\mathrm{i}J_z \tag{48}$$

$$[J_z, J_\pm]^- = \mp J_\pm. \tag{49}$$

The quantity

$$J^2 = J_x^2 + J_y^2 + J_z^2 = J_z^2 + \tfrac{1}{2}[J_+J_- + J_-J_+] \tag{50}$$

has vanishing brackets with all the generators of rotations in an irreducible realization. It is thus a Casimir which, in any irreducible canonical realization, is a constant [44], so that

$$[J^2, \xi_m] = \text{constant } \xi_m \equiv j(j+1)\xi_m. \tag{51}$$

Irreducible realizations ξ_m for definite m are thus more properly labelled by both eigenvalues j and m as ξ_{jm}.

The z-projection of the angular momentum of $[J_\pm, \xi_{jm}]$ is obtained by computing $[J_z, [J_\pm, \xi_{jm}]]$ with the aid of the Jacobi identity (20) and the bracket relations of the ladder operators from (49),

$$[J_z, [J_\pm, \xi_{jm}]] = -\mathrm{i}(m \pm 1)[J_\pm, \xi_{jm}]. \tag{52}$$

Continuing in this way, we may obtain the action of the ladder operators in the canonical formalism just as in quantum mechanics. In particular [43],

$$[J_\pm, \xi_{jm}] = -\mathrm{i}\sqrt{(j \mp m)(j \pm m + 1)}\xi_{jm\pm 1}. \tag{53}$$

It follows [43, 45, 46] that, as in quantum mechanics, realizations of integral and half-integral j occur. The eigenvectors of J_z given by (46) have integer or half-integer eigenvalues $-j \leqslant m \leqslant j$ and span a $(2j+1)$-dimensional invariant subspace of the Hilbert space of canonical realizations of the rotation subgroup. In what follows, ξ_{jm} will be treated collectively as components of an irreducible spherical tensor of rank j.

3.3. Classification of irreducible realizations of the Poincaré group

Irreducible canonical realizations of the Poincaré group may be classified in a manner entirely analogous to the method used in quantum field theory. Construct infinitesimal generators from the quantities defined in (37)–(39):

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{J} + \mathrm{i}\mathbf{K}) \tag{54}$$

and

$$\mathbf{B} = \tfrac{1}{2}(\mathbf{J} - \mathrm{i}\mathbf{K}). \tag{55}$$

These satisfy the bracket relations

$$[A_i, A_j]^- = \epsilon_{ijk}A_k, \qquad [B_i, B_j]^- = \epsilon_{ijk}B_k, \qquad [A_i, B_j]^- = 0. \tag{56}$$

The first two of these are identical to the Poisson brackets for $\mathbf{J}$. The ladder operator formalism developed in the preceding section therefore may be used to generate the components of an irreducible canonical realization in exactly the same way that a complete set of m values is generated for spin j. The irreducible canonical realizations are classified by a pair of indices (a, b), both of which may be either integral or half-integral. Generators A_3 and B_3 act on irreducible canonical realizations $\xi_{kl}^{(a,b)}$ according to

$$\left[A_3, \xi_{kl}^{(a,b)}\right] = -\mathrm{i}k\xi_{kl}^{(a,b)} \tag{57}$$

and

$$\left[B_3, \xi_{kl}^{(a,b)}\right] = -\mathrm{i}l\xi_{kl}^{(a,b)}, \tag{58}$$

where the eigenvalues

$$k = -a, -a+1, \ldots, +a \tag{59}$$

and

$$l = -b, -b+1, \ldots, +b. \tag{60}$$

A ladder operator for **A** is given by

$$\left[A_1 \pm \mathrm{i}A_2, \xi_{kl}^{(a,b)}\right] = -\mathrm{i}\sqrt{(a \mp k)(a \pm k+1)}\xi_{k\pm1,l}^{(a,b)} \tag{61}$$

and similarly for **B**.

Canonical realizations with a definite spin j are constructed from combinations of the (a, b) realizations. Thus, for example, a scalar field belongs to the $(0, 0)$ irreducible realization, while the Dirac field belongs to the $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ realization. Each such realization spans a $(2a+1)(2b+1)$ subspace of the Hilbert space of irreducible canonical realizations. Realizations with integral j are sometimes called *tensorial*, and those with half-integral j, *spinorial* [38]. Writing

$$J_i = A_i + B_i, \tag{62}$$

we see that J_3 will have eigenvalue $m \equiv k+l$ and J^2 will have eigenvalue $j(j+1)$ where $j = a+b$. Following Weinberg [47], an element of the (a, b) irreducible canonical realization is called a general causal field. In the following, once (a, b) is fixed, the field ξ will be written

$$\xi_{jm} \equiv \xi_{kl}^{(a,b)} \tag{63}$$

with

$$j = a+b \qquad m = k+l \tag{64}$$

or simply

$$\xi_m \equiv \xi_{jm} \tag{65}$$

unless it is necessary to specify the exact realization under discussion.

4. Parity and classical fields

4.1. Canonical realizations of the parity operation

The preceding section treated the effect of continuous coordinate transformations upon classical Grassmann fields required for the proof of the spin-statistics connection. A complete realization of the Poincaré group also includes the discrete transformations of parity, time reversal and charge conjugation. Classical analogues of charge conjugation and time reversal are discussed in [1]. The action of parity on a scalar function ξ of spacetime location $\mathbf{x}, t$ is

$$\mathcal{P}(\xi(\mathbf{x}, t)) = \eta\xi(-\mathbf{x}, t). \tag{66}$$

$\mathcal{P}$ commutes with the generators of time translations and rotations, but anticommutes with the generators of spatial translations and boosts. Pauri and Prosperi [35] show that the operator $\hat{P}$ that realizes $\mathcal{P}$ in a canonical realization of the Poincaré group has the action

$$\hat{P}(\mathbf{J}) = \mathbf{J} \tag{67}$$

and

$$\hat{P}(\mathbf{K}) = -\mathbf{K} \tag{68}$$

on the generator of infinitesimal canonical transformations for rotations and boosts, respectively. Equations (67) and (68) are to be understood as shorthand for relations of the form

$$[\hat{P}(\mathbf{Q}), \xi] = \pm[\mathbf{Q}, \xi], \forall \xi. \tag{69}$$

In particular (*vide* also (98) below), the action of $\hat{P}$ is diagonal on components of a field of definite spin in a spherical tensor basis, allowing us to write

$$\hat{P}(\xi_m(x)) = \eta \xi_m(x'), \tag{70}$$

where

$$\mathbf{x}' = -\mathbf{x} \qquad t' = t. \tag{71}$$

Because $\hat{P}^2 = 1$,

$$\eta^2 = 1, \tag{72}$$

$$\eta = \pm 1 \tag{73}$$

for any field.

In quantum mechanics, the parity of a state is a multiplicative quantum number. The classical statement of this property to be used in section 5 is that the parity of the scalar product of two fields is the product of the parities of the individual fields. Start with the observation that a scalar function of position has even parity under space inversion. A scalar is unchanged by any transformation of reference frame [48]. If one expresses a scalar field $\xi(\mathbf{r})$ in terms of a new coordinate frame $\mathbf{r}''$ as $\xi''(\mathbf{r}'')$ [49],

$$\xi''(\mathbf{r}'') = \xi(\mathbf{r}). \tag{74}$$

Fix a common origin for both $\mathbf{r}$ and $\mathbf{r}''$. The frame $\mathbf{r}''$ is assumed to differ from $\mathbf{r}$ by a rotation defined by Euler angles α, β, γ. Denote the action of the rotation that carries $\mathbf{r}$ to $\mathbf{r}''$ by the rotation operator $\mathcal{D}^{(1)}(\alpha\beta\gamma)$ and that which carries ξ to ξ'' by $\mathcal{D}^{(0)}(\alpha\beta\gamma)$:

$$\mathbf{r}'' = \mathcal{D}^{(1)}(\alpha\beta\gamma)\mathbf{r} \tag{75}$$

and

$$\xi'' = \mathcal{D}^{(0)}(\alpha\beta\gamma)\xi. \tag{76}$$

The rotation operator $\mathcal{D}^{(0)}(\alpha\beta\gamma)$ for a scalar is simply unity. By (74) we thus have

$$\xi(\mathcal{D}^{(1)}(\alpha\beta\gamma)\mathbf{r}) = \xi(\mathbf{r}) \tag{77}$$

for any rotation α, β, γ. Once an origin has been fixed, the value of ξ does not depend upon the orientation of the coordinate axes. Accordingly, ξ can depend only upon the magnitude of $\mathbf{r}$:

$$\xi(\mathbf{r}) = \xi(|\mathbf{r}|). \tag{78}$$

In particular,

$$\xi(-\mathbf{r}) = \xi(\mathbf{r}). \tag{79}$$

The parity of a scalar field is thus $\eta = +1$.

Next, the scalar product of two fields with the same rank is given by [50]

$$\xi \cdot \zeta = \sum_m (-1)^m \xi_m \zeta_{-m}. \tag{80}$$

Equation (80) is proportional to the expression for the coupling of ξ and ζ to spin zero [51], i.e., a scalar function of position. Consider the effect of the parity operation on the scalar product in (80),

$$\hat{P}(\xi(x)\cdot\zeta(x)) = \eta_{\xi\zeta}\xi(x')\cdot\zeta(x'). \tag{81}$$

With the aid of (70) and (73), this may be written as

$$\eta_{\xi\zeta}\hat{P}(\xi(x)\cdot\zeta(x)) = \eta_\xi\eta_\zeta\hat{P}(\xi(x))\cdot\hat{P}(\zeta(x)). \tag{82}$$

The only choice consistent with a nonvanishing scalar product is readily seen to be

$$\hat{P}(\xi(x)\cdot\zeta(x)) = \hat{P}(\xi(x))\cdot\hat{P}(\zeta(x)) \tag{83}$$

and

$$\eta_{\xi\zeta} = \eta_\xi\eta_\zeta. \tag{84}$$

4.2. Parity and general canonical realizations of the Poincaré group

Consider next the effect of space inversion on the generators **A** and **B** of (54) and (55). Recall from (67) and (68) that under the parity operation the generator of rotations is even, and that of boosts is odd, with the result

$$\hat{P}(\mathbf{A}) = \mathbf{B}, \qquad \hat{P}(\mathbf{B}) = \mathbf{A}. \tag{85}$$

Making use of the identity

$$\hat{P}([\xi,\zeta]) = [\hat{P}(\xi),\hat{P}(\zeta)], \tag{86}$$

which follows from the definition of a canonical transformation and the action of parity [35, 52], apply the parity operation to (61) to obtain [53]

$$\hat{P}\big(\big[A_\pm,\xi_{kl}^{(a,b)}\big]\big) = -\mathrm{i}\sqrt{(a\mp k)(a\pm k+1)}\,\hat{P}\big(\xi_{k\pm1,l}^{(a,b)}\big) \tag{87}$$

$$= \big[\hat{P}(A_\pm),\hat{P}\big(\xi_{kl}^{(a,b)}\big)\big] \tag{88}$$

$$= \big[B_\pm,\hat{P}\big(\xi_{kl}^{(a,b)}\big)\big]. \tag{89}$$

By the same reasoning

$$\big[A_\pm,\hat{P}\big(\xi_{kl}^{(a,b)}\big)\big] = -\mathrm{i}\sqrt{(b\mp k)(b\pm k+1)}\,\hat{P}\big(\xi_{k,l\pm1}^{(a,b)}\big) \tag{90}$$

From (87)–(90) we conclude

$$\hat{P}\big(\xi_{kl}^{(a,b)}(\mathbf{x},t)\big) \propto \xi_{lk}^{(b,a)}(-\mathbf{x},t). \tag{91}$$

Set $j = a+b$ and write

$$\hat{P}\big(\xi_{kl}^{(a,b)}(\mathbf{x},t)\big) \equiv \eta_{kl}\Phi(j,a,b)\xi_{lk}^{(b,a)}(-\mathbf{x},t) \tag{92}$$

with

$$\eta_{kl}^2\Phi^2(j,a,b) = 1. \tag{93}$$

Then

$$\hat{P}\big(\big[A_\pm,\xi_{kl}^{(a,b)}\big]\big) = -\mathrm{i}\sqrt{(a\mp k)(a\pm k+1)}\eta_{k\pm1l}\Phi(j,a,b)\xi_{lk\pm1}^{(b,a)}(-\mathbf{x},t) \tag{94}$$

$$= \big[B_\pm,\eta_{kl}\Phi(j,a,b)\xi_{lk}^{(b,a)}(-\mathbf{x},t)\big] \tag{95}$$

$$= -\mathrm{i}\sqrt{(a\mp k)(a\pm k+1)}\eta_{kl}\Phi(j,a,b)\xi_{lk\pm1}^{(b,a)}(-\mathbf{x},t). \tag{96}$$

Divide out common terms in (94) and (96) to obtain

$$\eta_{k\pm 1l} = \eta_{kl} \tag{97}$$

and likewise upon exchanging **A** and **B** in the foregoing,

$$\eta_{kl\pm 1} = \eta_{kl} \equiv \eta. \tag{98}$$

As before, $\hat{P}^2 \equiv 1$ allows us to write $\eta^2 = 1$, so

$$\hat{P}\big(\xi_{kl}^{(a,b)}(\mathbf{x}, t)\big) \equiv \eta\Phi(j, a, b)\xi_{lk}^{(b,a)}(-\mathbf{x}, t). \tag{99}$$

Although we make no use of it, the choice of Φ consistent with conventions for general causal fields in quantum field theory [54] is

$$\Phi(j, a, b) = (-1)^{(a+b-j)}. \tag{100}$$

5. Connection between spin and statistics

Before attacking the case of general causal fields, the method of proof is worked out for the simpler case of $(j, 0)$ representations [13]. Define the (Weinberg) field

$$\xi_m \equiv \xi_{m0}^{(j,0)}, \tag{101}$$

where m runs from $-j$ to j. It will be shown that imposing local commutativity on components of a field $\xi_m(\mathbf{x}, t)$ leads to the spin-statistics connection [55].

The central element of the proof relies on the scalar product of a certain field evaluated at one spacetime location with the same field, evaluated at a point lying at spacelike interval from the first. A Lorentz frame exists in which the two points occur at equal time. The fields may therefore be written as $\xi(\mathbf{y}+\mathbf{x}, t)$ and $\xi(\mathbf{y}, t)$, and the scalar product as

$$\xi(\mathbf{y}+\mathbf{x}, t) \cdot \xi(\mathbf{y}, t). \tag{102}$$

By translational invariance, this must be identical with

$$\xi(\mathbf{x}, t) \cdot \xi(\mathbf{0}, t) = \xi(\mathbf{x}/2, t) \cdot \xi(-\mathbf{x}/2, t), \tag{103}$$

which we write as

$$\xi(\mathbf{x}, t) \cdot \xi(-\mathbf{x}, t) \tag{104}$$

from here on. While this object can be regarded as a purely formal device, it is closely related to a quantity which finds use elsewhere is classical physics, the correlation function. The spatial autocorrelation of ξ,

$$\mathbf{g}(\mathbf{x}, t) = \langle \xi(\mathbf{y}+\mathbf{x}, t)\xi(\mathbf{y}, t)\rangle, \tag{105}$$

is of great significance in theories of statistical fluctuations [56, 57]. The angle brackets denote a spatial average over $\mathbf{y}$. While correlation functions find more use in classical statistical mechanics, where they are used to describe the relation between fluctuations in particle occupation number at distinct points in (say) an ideal gas, correlation functions appear in continuum physics as well, most notably in the theory of interference of electromagnetic fields [58, 59], but also in descriptions of phenomena as diverse as intensity interferometry in radio and optical astronomy [60], pressure fluctuations in fluid mechanics [61], fluctuation–dissipation relations in acoustics and electromagnetism [62, 63] and critical opalescence [64]. Tensor correlation functions also find use in the study of anisotropy and polarization of the cosmic microwave background [65].

Commence by disposing of the possibility that the quantity

$$\xi(\mathbf{x}, t) \cdot \xi(-\mathbf{x}, t) \tag{106}$$

might vanish identically for nontrivial fields. We do so by constructing a field for which (106) may be seen to be nonvanishing. From the scalar product of fields ψ and ϕ of the same rank form

$$(\psi, \phi) \equiv \frac{1}{2j+1} \int \mathrm{d}^3x \, \psi^*(\mathbf{x}, t) \cdot \phi(\mathbf{x}, t). \tag{107}$$

Equation (107) defines an inner product on the Hilbert space of complex square-integrable functions on a spacelike slice of R: (1) $(\psi + \eta, \phi) = (\psi, \phi) + (\eta, \phi)$ (2) $(a\psi, b\phi)$ is linear in b and antilinear in a. (3) $(\psi, \phi)^* = (\phi, \psi)$ (4) $(\psi, \psi) \geqslant 0$ and (ψ, ψ) vanishes iff ψ does. We may see this last as follows. In

$$(\psi, \psi) = \frac{1}{2j+1} \int \mathrm{d}^3x \, \psi^*(\mathbf{x}, t) \cdot \psi(\mathbf{x}, t), \tag{108}$$

we may expand the angular dependence of ψ in normal modes. The spin-weighted spherical harmonics ${}_sY_{jm}$ generalize ordinary spherical harmonics to arbitrary (to include half-integral) eigenvalues of dimensionless angular momentum [66–68]. The spin-weight s is the negative of a dimensionless helicity [69]. An irreducible realization ψ of spin j appearing in the integrand of equation (108) may be written as

$$\psi_m(r, \Omega) = f_j(r)\, {}_sY_{jm}(\Omega) \tag{109}$$

at radius r for some value of s. Because the canonical ladder operators that raise and lower m and s act on angular degrees of freedom only, the radial weight can have no dependence upon either m or s [70]. The harmonics satisfy

$${}_sY^*_{jm} = (-1)^{(s+m)}_{s} Y_{jm} \tag{110}$$

and

$$\int \mathrm{d}\Omega\, {}_sY^*_{jm'}\, {}_sY_{jm} = \delta_{m'm}. \tag{111}$$

It therefore may be arranged that

$$(\psi, \psi) = \int r^2 \,\mathrm{d}r \, f^*_j(r) f_j(r) \geqslant 0. \tag{112}$$

(One must watch the order of factors if ψ is odd.) In particular, if $\psi \cdot \psi^* = \psi^* \cdot \psi = 0$ almost everywhere, then $(\psi, \psi) = 0$. But $(\psi, \psi) = 0$ iff $\psi = 0$ a.e.

Assume ζ is a nonvanishing field belonging to a spin j irreducible canonical realization. From ζ form

$$\xi_m(\mathbf{x}, t) = \zeta^*_m(\mathbf{x}, t) \pm \zeta_m(-\mathbf{x}, t). \tag{113}$$

We have

$$\xi(\mathbf{x}, t) \cdot \xi(-\mathbf{x}, t) = \pm \xi(\mathbf{x}, t) \cdot \xi^*(\mathbf{x}, t). \tag{114}$$

As a general rule, the field ξ will have nonvanishing norm and the RHS of (114) will differ from zero. But suppose that for one choice of sign in (113), ξ were to vanish $\forall \mathbf{x}$. In that event ξ, and hence (114), cannot vanish for the other choice. We suppose in what follows that the appropriate choice of sign has been made, if necessary, and that (106) is therefore nonvanishing on some open set of $\mathbf{x}$.

The effect of $\hat{P}$ on the scalar product of $\xi(\mathbf{x}, t)$ and $\xi(-\mathbf{x}, t)$ is, according to (99) for $a = b = 0$,

$$\begin{aligned} \hat{P}(\xi(\mathbf{x}, t) \cdot \xi(-\mathbf{x}, t)) &= \hat{P}(\xi(\mathbf{x}, t)) \cdot \hat{P}(\xi(-\mathbf{x}, t)) \\ &= \xi(-\mathbf{x}, t) \cdot \xi(\mathbf{x}, t). \end{aligned} \tag{115}$$

This quantity is the product of two terms with the same parity and by (84) is even parity itself. Considered as a function of $\mathbf{x}$, an even parity scalar obeys $\hat{P}(f(\mathbf{x})) = f(\mathbf{x})$, thus we have

$$\xi(\mathbf{x}, t) \cdot \xi(-\mathbf{x}, t) = \xi(-\mathbf{x}, t) \cdot \xi(\mathbf{x}, t). \tag{116}$$

Commutation relations of a causal field ($-$ for Bose, $+$ for Fermi) vanish outside the light cone:

$$\xi_m(\mathbf{x}, t)\xi_n(-\mathbf{x}, t) \pm \xi_n(-\mathbf{x}, t)\xi_m(\mathbf{x}, t) = 0. \tag{117}$$

By (80)

$$\xi(-\mathbf{x}, t) \cdot \xi(\mathbf{x}, t) = \pm \sum_m (-1)^m \xi_{-m}(\mathbf{x}, t)\xi_m(-\mathbf{x}, t), \tag{118}$$

for Bose (+) or Fermi ($-$) fields, respectively. Invert the order of summation by replacing m with $-m'$:

$$\xi(-\mathbf{x}, t) \cdot \xi(\mathbf{x}, t) = \pm \sum_{m'} (-1)^{-m'} \xi_{m'}(\mathbf{x}, t)\xi_{-m'}(-\mathbf{x}, t). \tag{119}$$

Now, $j + m'$ is always an integer and $2j + 2m'$ an even integer. We may write

$$\begin{aligned}(-1)^{-m'} &= (-1)^{-m'}(-1)^{2j+2m'}\\ &= (-1)^{2j}(-1)^{m'}\end{aligned} \tag{120}$$

in (119) to obtain for (116)

$$\xi(\mathbf{x}, t) \cdot \xi(-\mathbf{x}, t) = \pm(-1)^{2j}\xi(\mathbf{x}, t) \cdot \xi(-\mathbf{x}, t), \tag{121}$$

or

$$1 = \pm(-1)^{2j}. \tag{122}$$

Equation (122) is a statement of the connection between spin and statistics.

The extension of the argument just given to the case of the general (a, b) representation is straightforward. The field $\xi^{(ab)}_{kl}$ now carries two indices $-a \leqslant k \leqslant a$ and $-b \leqslant l \leqslant b$, and (80) is replaced with an expression that couples two (a, b) spherical tensors to a $(0, 0)$ scalar, in a generalization of Racah's [51] original derivation of (80). That expression now becomes (retaining the dot product notation)

$$\begin{aligned}&\sum_{kl} \begin{pmatrix} a & a & 0 \\ -k & k & 0 \end{pmatrix} \begin{pmatrix} b & b & 0 \\ -l & l & 0 \end{pmatrix} \xi^{(a,b)}_{kl}(-\mathbf{x}, t)\xi^{(a,b)}_{-k-l}(\mathbf{x}, t)\\ &\qquad \propto \sum_{kl} (-1)^m \xi^{(a,b)}_{kl}(-\mathbf{x}, t)\xi^{(a,b)}_{-k-l}(\mathbf{x}, t) \equiv \xi(-\mathbf{x}, t) \cdot \xi(\mathbf{x}, t),\end{aligned} \tag{123}$$

where $m = k+l$ and the objects in parentheses are Wigner 3j symbols. It is readily shown that (83) and (84) are valid for the generalized scalar product, and that (123) vanishes iff $\xi(\mathbf{x}, t)$ does. By (99) for the (0,0) realization, the result of applying $\hat{P}$ to (123) once again gives (116). Both the spin j and summation index m are half-integral iff one of a and b is half-integral. Therefore, (121) and thus (122) hold for the general (a, b) realization, as well.

We conclude classical fields which are irreducible canonical realizations of spin j must be commuting, even Grassmann variables if j is an integer, and anticommuting, odd Grassmann variables if j is half-integral. From the symmetry properties of brackets given earlier follows immediately the conclusion that irreducible canonical realizations for integral j obey Poisson bracket relations, while realizations for half-integral j obey Poisson antibracket relations. If π^μ is the momentum conjugate to ξ_μ, then the brackets are

$$[\xi_\mu(t, \mathbf{x}), \xi_\nu(t, \mathbf{y})]^- = [\pi^\mu(t, \mathbf{x}), \pi^\nu(t, \mathbf{y})]^- = 0 \tag{124}$$

$$[\xi_\mu(t, \mathbf{x}), \pi^\nu(t, \mathbf{y})]^- = \delta(\mathbf{x} - \mathbf{y})\delta^\nu_\mu \tag{125}$$

for $2j =$ even, and

$$[\xi_\mu(t,\mathbf{x}), \xi_\nu(t,\mathbf{y})]^+ = [\pi^\mu(t,\mathbf{x}), \pi^\nu(t,\mathbf{y})]^+ = 0 \tag{126}$$

$$[\xi_\mu(t,\mathbf{x}), \pi^\nu(t,\mathbf{y})]^+ = -\delta(\mathbf{x}-\mathbf{y})\delta_\mu^\nu \tag{127}$$

for $2j =$ odd.

6. Comments

The result just proven may appear somewhat remote from typical problems encountered in applications of classical field theory. For one thing, fields in classical physics are generally constrained systems [25]. For another, if we except the special cases of the electromagnetic and gravitational fields, problems in classical physics involving canonical realizations of a definite, but otherwise arbitrary, value of dimensionless angular momentum should be uncommon. The connection might better be stated: fields described by tensorial canonical realizations obey fundamental Poisson bracket relations, while fields described by spinorial realizations obey fundamental Poisson antibracket relations.

While the approach taken in this paper is Poincaré invariant, the treatment is not manifestly covariant, in that it singles out spacelike slices. This difficulty is a familiar one in Hamiltonian treatments of problems with relativistic symmetry. It appears that the method of proof used in this paper could be cast in nonrelativistic language without a major change. The main difficulty would appear to be replicating the classification of irreducible realizations for general fields. For the special case of the classification of bound states of the hydrogen atom, it is possible to recapitulate the construction in section 3.3 in nonrelativistic terms by substituting the Lenz vector for the boost generator $\mathbf{K}$, [71] but this method is not available in general. It hardly seems necessary, however, to replicate every detail of the structure of irreducible realizations of the Poincaré group in a nonrelativistic treatment, so long as the rotation group is realized faithfully. The distinguished role of spacelike slices in the canonical formalism naturally poses no difficulty in a nonrelativistic setting.

However, it must be questioned whether such a 'nonrelativistic' proof could really be considered satisfactory. Elements of the present demonstration, such as equal-time commutativity of fields, the effect of space inversion, and rotational symmetry, that all follow from the single requirement of Poincaré invariance, would evidently enter a nonrelativistic version of the proof as distinct hypotheses. This hardly seems parsimonious. Moreover, it has been argued in a critique of proofs of the spin-statistics connection in nonrelativistic quantum mechanics that no nonrelativistic analogue of local commutativity exists [72]. Even if this objection be set aside, it does not seem that any real advantage is to be gained from a nonrelativistic formulation.

7. Conclusion

Simple arguments based upon a field-theoretical canonical treatment of rotational and space-inversion symmetry lead to a proof of the spin-statistics connection for classical Grassman fields which are irreducible canonical realizations of the Poincaré group.

References

[1] Morgan J A 2004 Spin and statistics in classical mechanics *Am. J. Phys.* **72** 1408–17
[2] Casalbuoni R 1976 On the quantization of systems with anticommuting variables *Nuovo Cimento* A **33** 115–24

[3] Casalbuoni R 1976 The classical mechanics for Bose–Fermi systems *Nuovo Cimento* A **33** 389–431

[4] Berezin F A and Marinov M 1977 Particle spin dynamics as the Grassmann variant of classical mechanics *Ann. Phys.* **104** 336–62

[5] Galvao C A P and Teitelboim C 1980 Classical supersymmetric particles *J. Math. Phys.* **21** 1863–80

[6] Gomis J Novell M and Rafanelli K 1986 Pseudoclassical model of a particle with arbitrary spin *Phys. Rev.* D **34** 1072–1075

[7] Di Vecchia P and Ravndal F 1979 Supersymmetric Dirac particles *Phys. Lett.* A **73** 371–3

[8] Goldstein 1950 *Classical Mechanics* (Reading, MA: Addison-Wesley) chapter 11

[9] Saletan E J and Cromer A H 1971 *Theoretical Mechanics* (New York: Wiley) chapter 8

[10] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley) chapter 10

[11] Wenztel G 1949 *The Quantum Theory of Fields* (New York: Wiley-Interscience)

[12] Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (New York: McGraw-Hill) chapter 1

[13] Weinberg S 1964 Feynman rules for any spin *Phys. Rev.* **133** B 1318–32

[14] Weinberg S 1995 *The Quantum Theory of Fields I* (Cantab: Cambridge University Press)

[15] Marsden J E, Montgomery R, Morrison P J and Thompson W B 1986 Covariant Poisson brackets for classical fields *Ann. Phys.* **169** 29–47

[16] Kastrup H A 1983 Canonical theories of Lagrangian dynamical systems in physics *Phys. Rep.* **101** 1–167

[17] Salmon R 1988 Hamiltonian fluid mechanics *Ann. Rev. Fluid Mech.* **20** 225–56

[18] Jackiw R, Nair V P, Pi S-Y and Poluchronakos A P 2004 Perfect fluid theory and its extensions *J. Phys. A: Math. Gen.* **37** R327–432

[19] Kupershmidt B 1985 Odd and even Poisson brackets in dynamical systems *Lett. Math. Phys.* **9** 323–30

[20] Kupershmidt B 1986 A review of superintegrable systems appendix 2 *Nonlinear Systems of Partial Differential Equations in Applied Mathematics* (*Lecture Notes in Applied Mathematics* vol 23) ed B Nicolaenko, D D Holm and J M Hyman (Providence, RI: American Mathematical Society)

[21] Gozzi E Rohrlich D Thacker W D 1990 Classical adiabatic holonomy in field theory *Phys. Rev.* D **42** 2752–62

[22] Floreanini R and Jackiw R 1988 Functional representation for fermionic quantum fields *Phys. Rev.* D **37** 2206–24

[23] Berezin F A 1966 *The Method of Second Quantization* (New York: Academic)

[24] Swanson M S 1992 *Path Integrals and Quantum Processes* (New York: Academic) chapter 5

[25] Kupershmidt B 1992 *The Variational Principles of Dynamics* (Singapore: World Scientific)

[26] Saletan E J and Cromer A H 1971 *Theoretical Mechanics* (New York: Wiley) chapter 8 p 290

[27] Berezin F A and Kac G I 1970 Lie groups with commuting and anticommuting parameters *Math. USSR Sb.* **11** 311–25

[28] Corwin L, Ne'eman Y and Sternberg S 1975 Graded Lie algebras in mathematics and physics (Bose–Fermi symmetry) *Rev. Mod. Phys.* **47** 573–603

[29] Weinberg S 2000 *The Quantum Theory of Fields III* (Cantab: Cambridge University Press) pp 25–6

[30] Saletan E J and Cromer A H 1971 *Theoretical Mechanics* (New York: Wiley) chapter 8 p 291

[31] Dirac P A M 1958 *The Principles of Quantum Mechanics* 4th edn (Oxon: Clarendon) p 60

[32] Wigner E 1939 On unitary representations of the inhomogeneous Lorentz group *Ann. Math.* **40** 149–204

[33] Weinberg S 1995 *The Quantum Theory of Fields I* (Cantab: Cambridge University Press) pp 58–62

[34] Pauri M and Prosperi G M 1966 Canonical realizations of Lie symmetry groups *J. Math. Phys.* **7** 366–75 1966

[35] Pauri M and Prosperi G M 1975 Canonical realizations of the Poincare group: I. General theory *J. Math. Phys.* **16** 1503–21

[36] Dirac P A M 1949 Forms of relativistic dynamics *Rev. Mod. Phys.* **21** 392–9

[37] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley) chapter 20

[38] Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley) pp 58–61

[39] Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (New York: McGraw-Hill) chapter 1 pp 28–30

[40] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley) chapter 10 pp 407–412, 488–496

[41] Currie D G, Jordan T F and Sudarshan E C G 1963 Relativistic invariance and Hamiltonian theories of interacting particles *Rev. Mod. Phys.* **35** 350–75

[42] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley) chapter 10 pp 352–355, 402–406, 497

[43] Loinger A 1963 New concept of representation of a Lie algebra *Ann. Phys.* **23** 23–7

[44] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley) Chapter 20, pp 321–8

[45] Dirac P A M 1958 *The Principles of Quantum Mechanics* 4th edn (Oxon: Clarendon Press) pp 144–6
[46] Tung W-K 1985 *Group Theory in Physics* (Singapore: World Scientific) pp 103–5
[47] Weinberg S 1995 *The Quantum Theory of Fields I* (Cantab: Cambridge University Press) pp 233–44
[48] Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley) p 94
[49] Edmonds A R 1960 *Angular Momentum in Quantum Mechanics* (Princeton, NJ: Princeton University Press) p 54
[50] Edmonds A R 1960 *Angular Momentum in Quantum Mechanics* (Princeton, New Jersey: Princeton University Press) p 72
[51] Racah G 1942 Theory of complex spectra II *Phys. Rev.* **62** 438–462, 942
[52] Kupershmidt B 1986 A review of superintegrable systems appendix 2 *Nonlinear Systems of Partial Differential Equations in Applied Mathematics (Lecture Notes in Applied Mathematics* vol 23*)* ed B Nicolaenko, D D Holm and J M Hyman (Providence, RI: American Mathematical Society) p 110
[53] Weinberg S 1995 *The Quantum Theory of Fields I* (Cantab: Cambridge University Press) pp 76–77
[54] Weinberg S 1995 *The Quantum Theory of Fields I* (Cantab: Cambridge University Press) pp 239–40
[55] Morgan J A 2005 Parity and the spin-statistics connection *Pramana J. Phys.* **65** 513–6
[56] Landau L D and Lifschitz E M 1958 *Statistical Physics* (Reading, MA: Addison-Wesley) pp 363–79
[57] Huang K 1987 *Statistical Mechanics* 2nd edition (New York: Wiley) pp 394–6
[58] Born M and Wolf E 1980 *Principles of Optics* 6th edn (New York: Pergamon) pp 499–505
[59] Mandel L and Wolf E 1965 Coherence of optical fields *Rev. Mod. Phys.* **37** 235–42
[60] Hanbury Brown R and Twiss R Q 1954 A new type of interferometer for use in radio astronomy *Phil. Mag.* **45** 663–82
[61] Landau L D and Lifschitz E M 1959 *Fluid Mechanics* (Reading, MA: Addison-Wesley) pp 523–9
[62] Weber J 1956 Fluctuation dissipation theorem *Phys. Rev.* **101** 1620–26
[63] Callen H B and Welton T A 1951 Irreversibility and generalized noise *Phys. Rev.* **83** 34–40
[64] Domb C 1996 *The Critical Point* (New York: Taylor and Francis) pp 124–6
[65] Ng K-W and Liu G-C 1999 Correlation functions of CMB anisotropy and polarization *Int. J. Mod. Phys.* D **8** 61–83
[66] Newman E T Penrose R 1966 Note on the Bondi–Metzner–Sachs group *J. Math. Phys.* **7** 2155–61
[67] Goldberg J N, Macfarlane A J, Newman E T, Rohrlich F and Sudarshan E C G 1967 Spin-s spherical harmonics and ð *J. Math. Phys.* **11** 863–70
[68] Hu W and White M 1997 CMB anisotropies: total angular momentum method *Phys. Rev.* D **56** 596–615
[69] Campbell W B 1971 Tensor and spinor spherical harmonics and the spin-s harmonics ${}_sY_{jm}$ *J. Math. Phys.* **12** 1763–70
[70] Baym G 1969 *Lectures on Quantum Mechanics* (Reading, MA: Benjamin) p 163
[71] Biedenharn L C 1961 Wigner coefficients for the R_4 group and some applications *J. Math. Phys.* **2** 433–41
[72] Allen R E and Mondragon A R 2003 No spin-statistics connection in nonrelatvistic quantum mechanics *Preprint* quant-ph/0304088